Journal of Statistical Physics, Vol. 47, Nos. 1/2, 1987

On the Dynamics of Simple Hierarchical Systems

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Received September 22, 1986; revision received December 5, 1986

Some applications of ultrametric and hierarchical structures are discussed in the context of condensed matter physics. It is shown that a simple arrangement of barriers leads to the Vogel–Fulcher law for the diffusion contant. Evidence is given that in this model the Vogel–Fulcher law is independent of the Kohlrausch–Williams–Watts law for relaxation.

KEY WORDS: Ultrametricity; dynamical phase transitions on hierarchical systems; Vogel–Fulcher law; relaxation.

1. INTRODUCTION

The behavior of complex disordered systems becomes a central point in condensed matter physics. The complexity of disordered systems is the main difficulty in the theoretical treatment of random matter. Despite these difficulties, much effort has been made. Examples include the spin glasses as typical candidates for a system with quenched disorder[(1)] and similar approaches exploring more complex questions, such as neural networks[(2)] or such optimization problems as the traveling salesman problem,[(3)] the matching problem,[(4)] or graph partioning.[(5)]

The step to "ordinary" glasses seems too large at the moment, but one can learn from the theory of spin glasses and try to construct simple models for highly complex random systems and see how far one can get. In order to give arguments for simple models, we mention here some of the most important features of complex systems. On the microscopic level we have disorder and frustration. In spin glasses this is well known.[(1)] In glasses frustration is not as simple as in magnetic systems, but by transfor-

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0022-4715/87/0400-0133$05.00/0

ming the structure of network glasses into a higher dimensional, curved space one can employ similar concepts.[6] Another common feature is the configuration space structure. For the Sherrington–Kirkpatrick spin glass it has been shown that the free energy has many locally stable states ($\sim e^N$) and the configuration space looks like a random many-valley picture. The strongest common features of all disordered systems are observed macroscopically. Most such systems show a freezing behavior upon measurement of macroscopic quantities or transport properties (such as diffusion constants or viscosities in real glasses or spin correlation functions in spin glasses). Upon quenching of glass-forming liquids, the structure of the liquid freezes and a picture using stroboscopic light of the structure at the quench time can be taken. The freezing phenomena are always accompanied by extremely slow dynamics and the relaxation of macroscopic quantities departs drastically from the classical Debye relaxation or exponential relaxation. Examples of these departures are power law relaxations [$\sim (t/\tau)^a$] or the famous Kohlrausch–Williams–Watts (KWW) form of relaxation, which is a stretched exponential [$\sim \exp[-(t/\tau)^\alpha]$, $0 < \alpha < 1$].[7]

Measuring the relaxation time τ, one always finds a strong increase of τ with the temperature T. The strong increase is often put in the form of the Vogel–Fulcher (VF) law $\tau \sim \exp[+A/(T-T_0)]$,[8] where A and T_0 are fit parameters. It is believed that the VF law is universal for all glass-forming systems. This is also supported by the fact that T_0 ("Vogel temperature") is obtained to be $T_0 \approx T_g - \Delta T$, where $10 < \Delta T < 50$ K. Here T_g is the actual glass or freezing temperature.[9] The same VF law is obtained for the diffusion coefficient and the viscosity. The above VF law is disputed by power laws from dynamical scaling and critical slowing down,[10] where τ is given by $\tau \sim (T-T_g)^{-\nu z}$, where ν is the correlation length exponent and z the dynamical scaling exponent. Experiments can be fitted by both laws reasonably well, but in the power law the exponent νz is very high for freezing processes (spin-glass transition).[11]

These brief introductory remarks show some difficulties in the dynamics of disordered systems, and we cannot solve them in this paper. The aim of this work is to take a simple model and try to find some of the macroscopic consequences discussed above.

Of course, there are many ways to produce the KWW law, so that the situation becomes more and more difficult in deciding if a model is valid or not (by means of experiments). Most of the models are summarized in Ref. 7.

The idea of using hierarchical constraints in the context of relaxation was put forward by Palmer *et al.*,[12] who assumed that, for example, a spin can only relax if several others relax, acting as a constraint to the first spin.

In this case the KWW law is a consequence of a specially chosen relaxation time distribution. As a consequence of their model, there have been discussions of the dynamics on ultrametric spaces, where it was assumed that the hierarchical constraints are ultrametric (see Ref. 13 for details). These authors have been able to show the occurence of various relaxation laws for special types of hierarchical barriers. They found power law relaxations and the KWW law. We will go into more detail later in the paper.

The situation of the VF law is not so well established. There are several phenomenological approaches of the free-volume type for real glasses[14] giving a VF law for the transport quantities. The VF law has been calculated for the "freezing" of a dense solution of hard rods by Edwards and Vilgis.[15] In this approach the VF law was a direct consequence of cooperative motion of several rods. One point we want to make in this paper is the possibility of finding a transition on a hierarchical structure that shows VF-type behavior. This result is an extension of recent work by Teitel and Domany[16] and Vilgis.[17]

2. ULTRAMETRIC SPACES AS A TOY MODEL

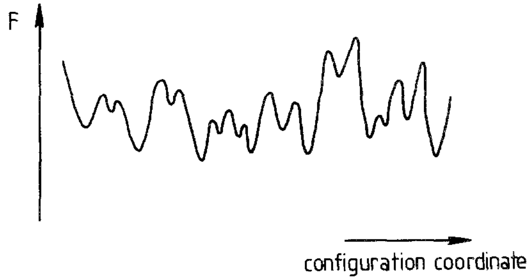
Ultrametricity entered into physics very recently (1984), when Mezard *et al.*[18] discovered that the pure states in a spin glass have ultrametric structure. Despite the fact that ultrametricity is very new in physics, it is very old in mathematics and classification. For an excellent review on these topics and also on biological applications see Ref. 19. A space is an ultrametric space if the distance between joints of the space satisfies the strong triangle inequality

$$d(A, C) \leqslant \max\{d(A, B), d(B, C)\} \tag{2.1}$$

where $d(A, B)$ is the distance between points A and B. Note that this is a stronger inequality than the usual triangular one.

How can we make use of the hierarchical model for some of the problems mentioned above? For further discussion, let us consider the free energy surface of a random system. For spin glasses the free energy surface contains e^N local minima (N is the number of spins). For glasses a similar number of minima seem to be present (in this case N is the number of atoms). If we assume that in the free energy surface there is no intrinsic scale and that it looks the same on many scales, we can assume that it will be self-similar on some scales. This means that there are valleys within valleys and so on.

To give an illustration, we suppose that the energy surface is given by a picture like that in Fig. 1.



If we enlarge one of the valleys we will find more or less the same picture as shown there. Suppose for further discussion that if one of the valleys is occupied (i.e., the state of the system is represented by one of the valleys and the system changes by crossing a barrier), it could only reach the next valley, e.g., Fig. 2.

This change in the configuration space by crossing barriers would mean in real space that a certain number of spins in a spin glass or a number of atoms in an ordinary glass would have to flip or to move until a new configuration is taken. The new configuration is described by the new valley. This can now mapped on the diffusion of a particle in such a potential. The particle represents the state of the system and the barrier height is a measure of the difficulty of changing the configuration of the system. Of course this argument is very handwaving and needs further investigation, but here this will be assumed.

The problem is now cooked down to the diffusion of a particle in a random potential, and since we assumed that only the topologically next valley can be reached by a jump, the model is quasi-one-dimensional.

The situation of the freezing system, that, for instance, the system cannot find another configuration during the observation time, corresponds in the model assumed here to a particle trapped in one of the valleys and unable to make a further jump out of it. The rate of change before reaching the "final frozen" state would correspond to relaxation of the system.

The problem of one-dimensional motion on a one-dimensional chain with random barriers was extensively studied in Ref. 20, but the

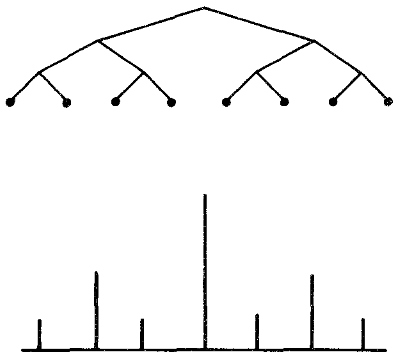


Fig. 3. Hierarchical arrangement of the barriers. The tree creates the energy barriers.

calculations are not trivial and correct averaging is not easy. By making use of the hierarchical order of barriers, it was possible in Refs. 13 and 21 to derive similar results as in Ref. 20 by much simpler methods. The hierarchical hypothesis says now that the random barrier chain can be replaced by a hierarchical arrangement of the barriers, as shown, for example, in Fig. (3). In this figure we have chosen the simplest regular scheme with two branches in the tree defining the barriers.[13] It is no problem to generalize this picture to, for example, random branching at each branching point, but this would not change the results drastically.[22]

3. DIFFUSION AND AUTOCORRELATION

Let us stick to some of the simplest possible models. The simplest model is that of a uniform branching index z all over the tree and the elementary height of the barriers is Δ (see Fig. 4). Further assumptions are that at all levels the barrier height increases linearly with the level of generation. The space we consider is now given by the bottom line of the tree. Imagine a particle sitting in one of the sites and wanting to jump to another site. For a long time after a sufficient number of jumps this would correspond to a random walk on a specially chosen structure. If we do not

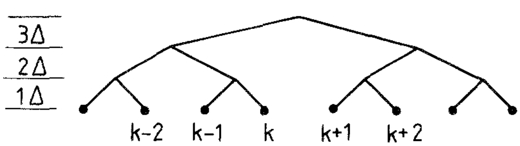


Fig. 4. A simple example of the arrangement of the barriers. The barrier height increases linearly with the level of generation. The branching index is 2 in this example.

have the constraint of quasi-one-dimensionality, the analysis is as carried out in Refs. 13 and 21 exactly by using master equations and transfer matrix methods. The analysis can be done by arguments given in Ref. 19 and we will repeat the argument as a remainder and for later use. We assume temperature-induced hopping and put for the transition probability

$$W \sim \exp[-\beta \Delta(m)] \tag{3.1}$$

where $\beta^{-1} = T$ is the temperature and $\Delta(m)$ is the energy barrier given at the mth level of generation. The simple argument now uses the effective time needed to move a distance m apart

$$t \sim \exp[\beta \Delta(m)] \tag{3.2}$$

The ultrametric distance m is used, which is defined here as the highest barrier that the particle has to cross. This implies that the particle can jump into a subcluster of the tree that is of ultrametric distance m away. In the simple ultrametric space given in Fig. 4 the particle can reach the mth cluster of the tree if it is able to cross the barrier of energy $\Delta(m)$; this is ruled by the probability proportional to the expression given in Eq. (3.1). It should be noticed that we have relaxed the constraint of quasi-one-dimensionality in this discussion and that no further (Euclidean) distance in addition to (non-Euclidean) ultrametric distance is present. This model is the pure model[19] and can be called ultrametric.

Obviously the first part of the problem is solved if Eq. (3.2) can be inverted with respect to m; then we find for the ultrametric distance m

$$m = \Delta^{-1}[1/\beta(\log t)] \tag{3.3}$$

where Δ^{-1} is the inverse function with respect to Δ [i.e., $x = \Delta^{-1}(\Delta(x))$]. In the Ogielski–Stein[13] case, where $\Delta(m) = \Delta m$, we find immediately

$$R(t) = m \sim (1/\Delta\beta) \log t \tag{3.4}$$

where $R(t)$ is the ultrametric distance the particle has traveled. Hence, we can only say in which cluster the particle has traveled, not the site (label k in Fig. 4). Equation (3.4) indicates an extremely slow anomalous diffusion with a strong temperature dependence. The term anomalous diffusion is used whenever deviations from normal diffusion $R(t) \sim t^{1/2}$ are found. The most popular example of anomalous diffusion is the ant[24] on a percolation cluster,[25] and it is present on each self-similar structure (fractals).[26]

The quantity responsible for relaxation is the autocorrelation function $P_0(t)$.[13] The quantity $P_0(t)$ in terms of random walks is given by considering a particle starting at the origin and measuring the time when it will come back to the origin. $P_0(t)$ can be calculated from the number of distinct visited sites explored by the particle in a time t.[27] This statement holds only for compact exploration, as clearly explained by de Gennes.[28] It also agrees with dynamical scaling for the probability function $P(x, t)$ if only one length matters in the problem. Then x is an arbitrary distance. The scaling hypothesis for $P(x, t)$ is then

$$P(x, t) = P_0(t)\ g(x/R(t)) \tag{3.5}$$

where $R(t)$ is the diffusion law and g a scaling function. Since the requirement of normalization of (3.5) holds, we find

$$P_0(t) \sim 1/[R(t)]^{d_f} \tag{3.6}$$

d_f is the fractal dimension of the lattice defined by the mass scaling with the distance. On a Euclidean lattice $d_f = d$ and $[R(t)]^d$ is the volume explored by the walker in a time t. In general, from (3.6) the spectral dimension can be defined if $R(t) \sim t^{1/d_w}$ follows a simple scaling law, d_w being the fractal dimension of the trajectory of the walk. It follows then from Eq. (3.6) that $P_0(t) \sim t^{-d_s/2}$, where d_s is the fracton—or spectral dimension (see Ref. 26 for details).

Application of these ideas to the ultrametric diffusion gives

$$P_0(t) \sim z^{-R(t)} \tag{3.7}$$

z is the branching index of the tree and $R(t)$ is given by Eq. (3.4). The final expression for $P_0(t)$ is then

$$P_0(t) \sim t^{-(\log z)/\beta\Delta} \tag{3.8}$$

which is now a power law with a linear temperature-dependent exponent. This is clearly nonexponential in time and agrees with the slow dynamics mentioned in the introduction.

Comparing this law with the general law on a self-similar structure $P_0(t) \sim t^{-d_s/2}$, we see that the pure ultrametric case would correspond to a continuously temperature-varying spectral dimension $d_s = 2T/\Delta \log z$. For $d_s > 2$ the walk becomes noncompact and $R(t) \sim t^{1/2}$ will be observed for $T > \Delta/\log z$.[29] This temperature $\theta = \Delta/\log z$ will play a crucial role in our further considerations.

A further remark should be made. The interpretation of the exponent in (3.8) as spectral dimension should be made more precise, since it would mean for the specific heat ($c \sim T^{d_s}$ in general) that a very peculiar temperature dependence would be observed,

$$C_{ums} \sim \exp[(2T/\Delta)(\log z) \log T] \tag{3.9}$$

which needs further comment, to be given elsewhere.

One limiting case pointed out by Ogielski and Stein[13] is the slowest rate of growth for the ultrametric structure that leads to a stable random walk $\Delta(m) = \Delta \log m$. By use of Eq. (3.3) it is found that

$$R(t) = m \sim t^{T/\Delta} \tag{3.10}$$

and for P_0

$$P_0(t) \sim \exp(-t^{T/\Delta} \log z) \tag{3.11}$$

which is the stretched exponential observed widely in experiments,[7] but here again with a temperature-dependent exponent, which means that the form of the relaxation changes with temperature. Here the transition temperature from normal to anomalous diffusion is at $T = \Delta$ [see Eq. (3.10)], so that for all $T > \Delta$ normal diffusion is expected rather than anomalous.

If we choose an arbitrary power law dependence of the states, we can apply the same arguments as above. Suppose now that the barriers follow a law like

$$\Delta(m) = \Delta m^{\alpha} \tag{3.12}$$

where α is an arbitrary power; we find

$$R(t) \sim (T/\Delta)(\log t)^{1/\alpha} \tag{3.13}$$

and for $\alpha \to \infty$ the diffusion becomes slower and slower, which is clear because the particle becomes more stuck in deeper and deeper valleys. By calculating the autocorrelation function $P_0(t)$, we find at once

$$P_0(t) \sim \exp[-(T/\Delta)(\log t)^{1/\alpha} \log z] \tag{3.14}$$

and we see that the choice of $\alpha = 1$ is the most peculiar one, since it will lead to a power law behavior in $P_0(t)$; for all other α, Eq. (3.14) holds. Later $\alpha = 1/2$ will correspond to a special case and here we find

$$P_0(t) \sim \exp[-(T/\Delta)(\log t)^2 \log z] \tag{3.15}$$

which is faster than a power law.

4. QUASI-ONE-DIMENSIONAL MODELS

There are some differences in the results if quasi-one-dimensional models are considered. The particle is now restricted to jump only to the next site and clearly this involves a Euclidean distance implicitly. This type of model is no longer purely ultrametric. Again we first summarize some results published earlier.[16,17] The finite dimensionality changes the result for the distance traveled from Eq. (3.4) to

$$R = z^m \tag{4.1}$$

and the effective jump probability

$$W \sim e^{-\beta \Delta(m)}(1/R) \tag{4.2}$$

including the entropy contribution for the particle restricted to jump only to the topologically next site. From $t \sim 1/W$ we readily obtain[19]

$$R(t) \sim t^{\frac{T \log z}{T \log z + \Delta}} \tag{4.3}$$

if it is assumed that $\Delta(m) = \Delta m$.

In this case $P_0(t)$ can be derived by a simple argument, and because the model is one-dimensional, $P_0(t) \sim 1/[R(t)]^1$ holds and

$$P_0(t) \sim t^{-\frac{T \log z}{T \log z + \Delta}} \tag{4.4}$$

This result has been confirmed in Refs. 16 and 30 by more sophisticated methods. There is again a transition from normal to anomalous diffusion.

As long as $P_0(t) \sim t^{-\rho}$ with $\rho < 1/2$ we find anomalous diffusion, since $p_0(t) \sim t^{-1/2}$ corresponds to the normal one-dimensional random walk. Here the transition occurs at the temperature

$$\theta = \Delta / \log z \tag{4.5}$$

so that for $T > \theta$ normal diffusion takes place and for $\theta > T$ anomalous diffusion will dominate the picture.

For further models, for instance, $\Delta(m) = \Delta \log m$, there are no simple expressions and $P_0(t)$ has to be calculated numerically. The reason is again the one-dimensional character, because of which we cannot invert the equations.

5. THE TRANSPORT AND RELAXATION QUANTITIES IN THE QUASI-ONE-DIMENSIONAL MODEL

It would be interesting to compute the transport coefficients under the conditions of the transition from normal to anomalous diffusion. Since in all the above considerations the transition takes place at a certain temperature, we expect a strong dependence of the transport coefficients on the temperature. The first step was taken by Teitel and Domany.[16] Their result can be derived along the same lines as the simple argument presented above.

We wish to calculate the diffusion constant as a function of temperature and start from the general result[30]

$$1/D = \langle 1/W \rangle \tag{5.1}$$

where $\langle 1/W \rangle$ is the averaged barrier height. The rigorous calculation would start from a master equation for the diffusing particle on the lattice. The barriers from site to site are explicitly in the master equation, and the diffusion constant can be calculated from the properties of the master equation, as shown, for example, by Zwanzig.[30]

Indeed, we can derive the same results by using the simpler arguments presented in the previous sections.

We calculate the diffusion constant at the level m for the quasi-one-dimensional model by the elementary definition

$$D = \lim_{t \to \infty} \frac{R^2(t)}{t} \tag{5.2}$$

where $R^2(t)$ is the mean square displacement and t is the time. Using Eqs. (4.1) and (4.2), we see that $R^2(t) = (z^m)^2$ and $t \sim W_{\text{eff}}^{-1} \sim z^m e^{\beta \Delta(m)}$. We find then for the effective diffusion constant in the mth level of generation

$$D_m = z^m e^{-\Delta(m)/T} \tag{5.3}$$

Equation (5.1) suggests we consider the inverse of D; summing over all effective inverse D_m, we obtain

$$1/D \sim \sum_m \exp[\Delta(m)/T - m \log z] \tag{5.4}$$

giving for the elementary model $\Delta(m) = \Delta m$

$$D \sim (1 - \theta/T) \tag{5.5}$$

where the transition temperature θ is given by

$$\theta = \Delta/\log z \tag{5.6}$$

The diffusion constant vanishes linearly with the ratio θ/T if the temperature is decreased, signaling again the transition from normal to anomalous diffusion. The physical interpretation of this law is that at high temperature the particle does not feel the barriers at all and its diffusion is according to the classical Einstein law $R^2 \sim t$. With lowered temperature the barriers become more and more important and at the transition temperature $\theta = \Delta/\log z$ anomalous diffusion sets in and the mean square distance traveled becomes extremely small compared to the Einstein case.

The question now is whether we can find a hierarchical set of barriers that turn the result (5.4) into the widely used Vogel–Fulcher law. For the diffusion this would mean where A is some constant $D \sim \exp[-A/(1-\theta/T)]$. It turns out that this is possible if the change

$$\Delta(m) = \Delta_0 m + \varepsilon \sqrt{m} \tag{5.7}$$

is made,[17] where ε is treated as a small correction compared to Δ_0 (this is not a major restriction, as we will see later). A physical motivation for Eq. (5.7) is, for instance, a small fluctuation of the barrier Δ, $\Delta = (\Delta_0 + \delta)$, where δ obeys a Gaussian probability distribution with a variance depending on the level m.[32] To be more general, we assume

$$\Delta(m) = \Delta_0 m + \varepsilon m^{1-\alpha} \tag{5.8}$$

where $0 < \alpha < 1$.

The diffusion constant is then calculated to be

$$\frac{1}{D} \sim \sum_{m=1}^{\infty} \exp\left(\frac{\Delta_0}{T} m + \frac{\varepsilon}{T} m^{\alpha} - m \log z\right) \tag{5.9}$$

The sum can be converted into an integral, which is evaluated by steepest descent; the result is given by

$$D \sim \exp\left[-(\log z)^{1-1/\alpha} (1-\alpha)^{1/\alpha} \left(\frac{\varepsilon}{T}\right)^{1/\alpha} \left(\frac{1}{1-\theta_0/T}\right)^{1/\alpha - 1}\right] \tag{5.10}$$

which is indeed of the VF type. The classical VF law corresponds to the choice of $\alpha = 1/2$,

$$D \sim \exp\left[-\frac{(\log z)^{-1}}{4} \left(\frac{\varepsilon}{T}\right)^2 \frac{1}{(1-\theta_0/T)}\right] \tag{5.11}$$

where $\theta_0 = \Delta_0/\log z$.

Let us now study the relaxation behavior of this particular model with $\alpha = 1/2$. Again we use the simple arguments from the previous sections and all we have to do use the relationship

$$t \sim \exp[\Delta(m)/T + m \log z] \tag{5.12}$$

where $\Delta(m)$ is given by Eq. (5.8) with $\alpha = 1/2$. Taking the logarithm of both sides of (5.12) and inserting Eq. (5.8), we find

$$\log t = (\Delta_0/T + \log z)\, m + (\varepsilon/T)\sqrt{m} \tag{5.13}$$

which is now a quadratic equation in $\sqrt{m}$. The general solution is given by

$$\sqrt{m} = \frac{1}{2}\frac{\varepsilon}{T}\frac{1}{\Delta_0/T + \log z}\left\{\left[1 + 4(\log t)\frac{T^2}{\varepsilon^2}\left(\frac{\Delta_0}{T} + \log z\right)\right]^{1/2} - 1\right\} \tag{5.14}$$

where the unphysical second sign has been dropped. The general mean displacement is given by use of Eq. (4.1); we find

$$\begin{aligned} R(t) &= z^m \\ &= \exp\left((\log z)\frac{1}{2}\left(\frac{\varepsilon}{T}\right)\frac{1}{\Delta_0/T + \log z}\right. \\ &\quad \times \left.\left\{\left[1 + 4(\log t)\left(\frac{T}{\varepsilon}\right)^2\left(\frac{\Delta}{T} + \log z\right)\right]^{1/2} - 1\right\}\right) \end{aligned} \tag{5.15}$$

and the autocorrelation function turns out to be

$$P_0(t) \sim R^{-1}(t) \tag{5.16}$$

because of the one-dimensionality.

$P_0(t)$ looks very complicated, but two limiting cases can be worked out. First, we consider the limit $\varepsilon \to 0$. In this limit the square root in Eq. (5.14) is dominated by the $1/\varepsilon^2$ term and we find

$$m = \log t/\Delta/T + \log z \tag{5.17}$$

so that the result quoted above [Eqs. (4.2) and (4.3)] is recovered, and $P_0(t)$ follows a power law decay.

Second, we consider $\varepsilon \to \infty$, so that the square root can be expanded and $P_0(t)$ obeys in this case

$$P_0(t) \sim \exp[-(T/\varepsilon)^2 (\log t)^2 \log z] \tag{5.18}$$

This formula agrees with that in Section 3.

Using the choice of the barriers given in (5.8), we find a Vogel–Fulcher law, but we do not find a Kohlrausch–Williams–Watts law for the relaxation in this one-dimensional model. This result would suggest that for this model both laws are independent and they are *not connected in general.*

6. SOME OTHER MODELS

We are now in position to discuss more models of the same type. We remark briefly on some possibilities. Choose, for example, the logarithmic arrangement of the barriers in the one-dimensional case. In this case the diffusion constant is given by

$$1/D \sim \sum_m \exp(\Delta/T \log m - m \log z) \tag{6.1}$$

Again we approximate the sum by the integral

$$1/D = \int dm\, m(\Delta/T)\, e^{-m \log z} \tag{6.2}$$

which can be estimated by the gamma function. Therefore, D becomes

$$D \sim (\log z)^{1+\Delta/T}[1/\Gamma(\Delta/T+1)] \tag{6.3}$$

and we find no transition temperature larger than zero. Clearly, if T approaches zero, $D \to 0$ as the Γ-function goes to infinity.

If the linear and the logarithmic models are mixed, we find a power law for D. To see this, suppose we have

$$\Delta(m) = \Delta_0 m + \varepsilon \log m \tag{6.4}$$

The diffusion constant is then calculated to be

$$D \sim (1 - \theta_0/T)^{1+\varepsilon/T} \tag{6.5}$$

and the exponent is a function of the temperature.

Finally, let us make some remarks on pure ultrametric models: All the transition temperatures should be lowered or even supressed. It is possible to derive similar formulas as discussed above, but they are not reported here.[32]

7. CONCLUSION AND DISCUSSION

This paper has discussed some properties (mainly on one-dimensional models) of structures with hierarchically distributed barriers. That these concepts are useful in condensed matter physics has recently been shown by various authors.[12,13,16,21] Clearly, the models are very simple and cannot give answers to all the questions raised in the introduction, but they might be a small step in understanding the properties of complex systems. In this paper the analysis is kept as simple as possible; detailed considerations need more refined mathematical methods. We have dealt here only with the simplest possible form of presenting the material, but of course we are aware that more advanced methods would give further results; for example, we have not discussed possible additional long-time behavior of the diffusion constant (see Ref. 31). Further, we cannot decide with these models whether the diffusion constant (and the inverse relaxation time $D \sim \tau^{-1}$) obeys a VF or a power law.

On the other hand, we have derived the VF law by a very simple assumption on the form of barriers. The physical reason for this assumption could be a special distribution of the barriers, which is not purely random as in Ref. 22, where it has been shown that for a purely random arrangement of the Δ's the results given in Section 3 are not altered drastically. If we assume that the fluctuating part of the Δ's obeys a distribution that depends on the level of generation (m), Eq. (5.7) can be motivated.[32]

One of the main results we want to stress is that we do not find a connection between the VF and the KWW law in our model, in contrast to other authors.[12] Hence, while Eq. (5.6) suggests a VF law for the diffusion constant (or the relaxation time), according to Eq. (5.14), we do not expect a drastic change in the autocorrelation function as long as ε is small.

A similar fact has been found recently in discussing a solution of dense, inflexible hard rods.[15] There the picture was that one rod is only able to move by translation (along its main axis), because the motion of the rod involving rotational degrees of freedom is hindered drastically by the presence of all the other rods. Then the rod can only progress if no barriers (=another rod) is in its way. If there is a barrier, this rod has to move out of the way, so the test rod can move. But in general the rod acting as a barrier is hindered by other rods, too, so that a whole barrier of rods has to move out of the way to enable the test rod to move. By incorporating cooperativity into this model by saying that collective motion is possible if all the barrier rods and the test rod are moving in a closed circle, a VF-type law was calculated. It was shown that the cooperativity does produce the VF law, but is not necessary to establish the KWW law.[22] The discussion in this paper seems to confirm this, despite the fact that we did not interpret Eq. (5.11) as a result of cooperativity and we used a different motivation than in Ref. 33.

A further point should be discussed. We did not mention the role of θ_0 in our derivation of the VF law. Here it is only the temperature where the transition appears. However, for real systems the VF law $\{\exp[-A/(T-T_0)]\}$ shows a singularity at T_0 somewhat below the glass temperature T_g. As mentioned in the introduction, an empirical rule is $T_0 \simeq T_g - 20$–50 K.[9] The role of T_0 is still not clear. An attempt has been made to connect T_0 with the so-called Kauzmann paradox, but the free volume interpretation seems to be more accepted (see Ref. 14 and references therein). For a more detailed discussion on the role of the freezing point and the temperature appearing in the VF law, see Ref. 33. The simple hierarchical model in this paper is, however, too simple to give further elucidation concerning this point.

One motivation for using the hierarchical structure is the assumption that the free energy surface is self-similar. Clearly this assumption remains to be proven.

Finally, our simple mapping of a diffusing particle on a complex line and the changes of state of a complex system are not obvious at first glance and need further consideration. Nevertheless, the behavior of particles on these hierarchical structures is interesting for further discussion. These kinds of structures open many further applications and problems not discussed in this paper. For example, it would be interesting to consider trees with various loops in it in order to look for a dependence of the results on the number of loops. One can think also of models with a temperature-dependent branching index so that at high temperatures there are more channels than at low temperatures.[32]

As a last remark, we point out again that similar results can be obtained for systems with random barriers (at least in one dimension), but in a much more difficult way.

ACKNOWLEDGMENTS

Helpful discussions on various parts of the paper with A. Blumen, A. Hernandes-Machado, K. L. Ngai, J. Souletie, and D. L. Stein are gratefully acknowledged.

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